# Matching Markets

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## I. Introduction

In the economic world money is a powerful player. The people who have it tend to fare better. In some situations the use of money can be unethical, and also unlawful. Let us think about the market for organs. In the typical supply and demand market where there is a shortage in organs, highest paying customers would be the only individuals who could obtain organs. How do we effectively and efficiently remedy this extreme health issue? How does the real world allocate organs?

According to Adam Smith's theory of markets, allocation will be a result of the "invisible hand." This is generally very true but is solely contingent on the premise of stability. If individuals could trade and become better off, then the "invisible hand" would not be leading to market equilibrium and the allocation is inherently unstable. In essence, stability requires that individuals cannot become better off from further trade. This is the crux of cooperative game theory, and is the idea on which we base our understanding of Matching Markets.



Matching Markets examples include students with Universities, Medical School Graduates with hospitals, organ donors with the sick in need, and students and thesis advisor. The important question to answer is how do stable matches occur? Agents participating care about whom they are matched with. There are two types of matching: two-sided one-to-one matching, and many-to-one matching. Two-sided one-to-one matching is the model that is more readily applicable to the real world and is what we will be discussing throughout the entirety of this chapter

## II. Model

A **set** is a group of agents. Sets distinguish the two types of players. There are two sets of players in matching markets. In this example we will call them I and J. For example in the market for thesis advisors, students would be in one set and thesis advisors in the other. There can be many individuals in sets or only 1, it depends completely on the scenario. Sets would be denoted as I={$i_1, i_2, \ldots, i_n$} and J={$j_1, j_2, \ldots, j_n$}, where each $i_n$ and $j_n$ are agents. Each $i$ has preferences over the J set, and each $j$ has preferences over the I set.

> For example, $i_2$ might have a *preference ordering* as such: P($i_2$)= $j_2, i_2, j_3, j_1$. This indicates that $i_2$ would choose to be matched with $j_2$ first, their second choice would be to remain unpaired (match with themselves, $i_2$), their third choice would be $j_3$, and their last choice would be $j_1$.

P is the set of preferences. Agents can have *strict* preferences, which were exemplified in the example above, however they can also have *indifferent* preferences. An agent in I is acceptable to an agent in J if agent *i* would prefer being with *j* to being unmatched. A player can be indifferent between two choices where they will not prefer one choice over another. This can be shown by putting brackets around the choices: i.e. P($i_2$)= $j_2$, $i_2$, [$j_3$, $j_1$]. This indicates $i_2$ is indifferent between $j_3$ and $j_1$ as their third choice. Their set of preferences is a complete ordering; meaning $i_2$ has a preference over every possible option. In this model we assume players are rational and therefore a player would always choose their best option. Preferences have to be complete and transitive. We assume preferences are transitive. I.E. If player $i_2$ prefers $j_3$ *to* $j_1$ and $i_2$ to $j_3$ then they prefer $i_2$ to $j_1$. All preferences are common knowledge.

**Preference Notation**

$J^1 \succ_i j^2$ means that i prefers $j^1$ to $j^2$

$J^1 \succ_i i$ means that i prefers $j^1$ to remaining single (matching with themselves), and

$i \succ_i j^1$ means that $j^1$ is unacceptable to i

A **Preference Table** is a visual listing the preferences of each agent.

| | 1 | 2 | 3 |
|---|---|---|---|
| A: | D | E | F |
| B: | E | F | D |
| C: | F | E | D |
| D: | B | C | A |
| E: | A | B | C |
| F: | A | C | B |

In the preference table above, there are two sets, one that contains A, B, and C and the other that contains D, E, and F. Each agent in the sets have preferences ranking from first to third over agents in the other sets. For example, P(A)=D, E, F; where D is the first choice, E the second, and F the third.

Each player has their preference ordering over other players, with who want to match with. Players ideally want to match with their first preference. However a match is made off of mutual preferences. They have to both prefer each other in order to have match. To see how players match, first we must understand the concept of a match.

A **match** consists of two agents from separate sets I and J: one *i* and one *j* make up a match. A matching is denoted as μ. A matching between *i* and *j* would mean there are two matches: μ(*i*)=*j* and μ(*j*)=*j*. *I* is matched to *j* and *j* is matched to *i.* There can be matching where an individual matches with themselves: μ(*i*)=*i*. For example, the following is a set of matched pairs:

$$\mu = \begin{array}{ccccc} j_2 & j_1 & j_4 & (i_4) & j_3 \\ i_1 & i_2 & i_3 & i_4 & i_5 \end{array}$$

This means $j_2$ is matched with $i_1$, thus μ($j_2$)=$i_1$ and μ($i_1$)=$j_2$. This is the same for each of the other matches, except for the fourth which claims that μ($i_4$)=$i_4$, corresponding with $i_4$'s preference to remain single.

Suppose there are 2 sets, who both have 3 agents, I={$i_1$, $i_2$, $i_3$} and J={$j_1$, $j_2$, $j_3$}. The matching M would pair ($i_1$, $j_1$) ($i_2$, $j_2$) and ($i_3$, $j_3$). M admits a **blocking pair** if any $j_N$ or $i_N$ prefer each other to their match in M. For example if $i_2$ preferred $j_3$ to their current match with $j_2$, and $j_3$ preferred $i_2$ to their current match with $i_3$, ($i_2$, $j_3$) would be a blocking pair because it disrupts two matches in M. For a match to be **stable** there cannot be a blocking pair.

Another criterion for stability is that every individual ends up being matched with an acceptable partner at the end of the algorithm. In order to find a stable match, there has to be a stable one-to-one matching. The game must have a finite number of rounds and all possible matchings have to be individually rational. For example, take I to be a set of men and J to be a set of women. Suppose I's and J's want to be matched together. $i_1$ prefers $j_2$, therefore the man $i_1$ went out and proposed to $j_2$ but was rejected. This means that at that moment $j_2$ preferred her unsettled match to $i_1$. $i_1$ then proposes to $j_2$ who accepts his proposal. As the game continues $j_2$ will look to do better and will inevitably in the end prefer her final match to $i_1$. There are no blocking pairs in this game and therefore this game is considered stable.

A stable matching produces a fixed pair. Think of a fixed pair as an unbreakable contract. A perfect matching is one where agents are all matched up at the end of the algorithm. This can only occur if there are an equal amount of players on each side to choose from so that each player is paired up with another and no two players are paired up with the same player.

A matching in the **core** of a two-sided one-to-one matching game has the property that no player can deviate from the "match" without disrupting the benefit it has to all players. This means that one player might be able to benefit more, but the other players will be hurt. The group as a whole suffers from this departure from the match. A match in the core must satisfy the following conditions: a. each player would rather have their partner than be single, b. each pair $(i, j)$ $i$ prefers $\mu(i)$ to $j$, and $j$ prefers $\mu(j)$ to $i$. This is true because if any player can improve on $\mu$ by staying single or if some pair of players can improve upon $\mu$ by matching with each other then they aren't in the core scenario. A matching in the core is essentially a stable match.

III. Active versus Passive

Within a matching scenario there is an **active** side and a **passive** side. The active side in one of the sets of agents, say I, and the passive side would therefore be the other set, J. The active side is the instigator of a match and proposes the match to the passive side. The passive side can reject or accept the matching. Passive side agents cannot initiate. Think of this like a marriage proposal. Say a man is the active side who proposes marriage to a female, the passive side. The female has the option to accept or reject the marriage offer. This is the common model of a matching market, however a unique model called deferred acceptance has an extra option for the passive side, which will be discussed in the later section. Outcomes (or matches) are dependent on who is the active and passive side and what their preferences are. The active side always has the upper hand because they have the unique ability to initiate a match and always come out better off than the passive side.

IV. Deferred-Acceptance Algorithm

The Deferred Acceptance Solution Concept, also known as the Gale-Shapley Algorithm, offers a way to always find stable matches. Simply, the passive side holds judgment on the active side's offer to match until the end of the algorithm. The idea behind it is that the matching proposals are made by a player in one set, say I, the active side. Each player i in I wants to match with a player in J. Each *i* proposes to the first *j* in their preference ranking. Each *j* who receives more than one proposal rejects all but the best of the proposals and retains the best (but does not actually accept it). This is like a promised engagement, but there are no strings attached. The *j*'s do not accept any proposal till the end of the algorithm, hence the name deferred-acceptance. The j's are able to keep their best available i's engaged, without having to accept them immediately. The rejected *i*'s then propose to the second in their preference ranking. These proposals are weighted again by each *j* who has received one and are rejected accordingly. This process continues until there are no more rejected proposals, which constitutes the end of the algorithm, and all matches are accepted. The procedure stops because there are a finite amount of players. At the end, no one holds a pair that is unacceptable.

**Solution Concept 0**: Deferred Acceptance (Gale-Shapley Algorithm)

The passive side holds judgment on the active side's offer to match until the end of the algorithm. This always produces a stable matching. It is a simple way to get rid of blocking pairs while producing stable matches.

**Theorem 1**: Always has an end result. There will always be resultant matches.

**Theorem 2**: Marriages produced by this algorithm are stable.

**Theorem 3**: The active side is paired with their best possible choice.

**Theorem 4**: The passive side is paired with their worst possible choice.

V. Practical Example

The following is an example, which will hopefully further your understanding of the matching markets.

Suppose there are 3 students who all need to be paired with their own thesis advisors. S={S1, S2, S3}, T={T1, T2, T3}. Each thesis advisor can only take on one student per semester. Suppose the preferences for both the students and the thesis advisors are as follows.

| | 1 | 2 | 3 |
|---|---|---|---|
| S1: | T2 | T1 | T3 |
| S2: | T2 | T3 | T1 |
| S3: | T1 | T2 | T3 |
| T1: | S1 | S2 | S3 |
| T2: | S3 | S1 | S2 |
| T3: | S2 | S1 | S3 |

This Preference Table indicates the complete preferences of all the agents in each set. We can list them out as follows:

P(S1)=T2, T1, T3
P(S2)=T2, T3, T1
P(S3)=T1, T2, T3
P(T1)=S1, S2, S3
P(T2)=S3, S1, S2
P(T3)=S2, S1 S3

Lets review some notation…

$T2 \succ_{S1} T1$, which means that player S1 prefers T2 to T1.

$T1 \succ_{S1} T3$, which means that S1 also prefers T1 to T3. Recall that preferences are transitive, so if S1 prefers T2 to T1 and T1 to T3 then they must prefer T2 to T3.

We can also note that there is no such preference where an individual would like to remain unmatched, or in other words, single. If an individual would prefer to remain unmatched, that would have been included in their preferences. With this being true, we can say the each of the three students would rather be matched with any thesis advisor than to remain single. This is also true for the thesis advisors, who would rather advise any of the three students than to remain without an advisee. The notation for this is as follows: $S_N \succ_{TN} T_N$ and $T_N \succ_{SN} S_N$.

To go about finding the stable matches, we must first know who is the active side. Lets say that the students must approach thesis advisors and ask them if they can be their advisee. This would thus make the thesis advisors the passive side. S1, S2, and S3, would clearly go ask their number one choice first. Therefore, S1 would ask T2, S2 would ask T2 and S3 would ask T1.

| | 1 | 2 | 3 |
|---|---|---|---|
| S1: | **T2** | T1 | T3 |
| S2: | **T2** | T3 | T1 |
| S3: | **T1** | T2 | T3 |
| T1: | S1 | S2 | **S3** |
| T2: | S3 | **S1** | **S2** |
| T3: | S2 | S1 | S3 |

As you can see, T2 was asked by two students to be their advisor. T2, having the option between his second and third choice, would obviously choose his second, which is S1. He would then reject S2 and retain S1. He would not directly promise to be S1's advisor but he would consider them as a tentative match, contingent upon his future offers. T1 would also enter into a tentative match with S3, because he is happy to at least be matching with a student, regardless if they are in fact his last choice. There would be two tentative matches after this first round of "proposals:" (S1,T2) (S3, T1). S2 is unmatched after this first round, and would therefore want to ask another advisor if he/she would be willing to match with them. S2 would go to their second choice now that their first choice rejected them. In this second round, S2 would ask T3 to be their advisor.

| | 1 | 2 | 3 |
|---|---|---|---|
| S1: | **T2** | T1 | T3 |
| S2: | ~~T2~~ | **T3** | T1 |
| S3: | **T1** | T2 | T3 |
| T1: | S1 | S2 | **S3** |
| T2: | S3 | **S1** | ~~S2~~ |
| T3: | **S2** | S1 | S3 |

As you can see, T3 would enter a tentative match with S2 because that is their best possible option. All players are matched; therefore this marks the end of the algorithm. The matches are as follows:

$\mu(S1)=T2$, $\mu(S2)=T3$, $\mu(S3)=T1$, $\mu(T1)=S3$, $\mu(T2)=S1$, $\mu(T3)=S2$

(S1, T2) (S2, T3) (S3, T1)

This is a stable match because

i. every student is matched with an acceptable thesis advisor, and every thesis advisor is matched with an acceptable student, and
ii. there is no possible way for a player to unilaterally deviate from these matches and be better off (or in other words: there is no blocking pair).

VI. Summary

There are two sets agents in matching markets, lets call them I and J. $I=\{i_1, i_2, ..., i_n\}$ and $J=\{j_1, j_2, ..., j_n\}$. Agents in sets have rational preferences (P) over whom they want to be matched with. Preferences are complete and transitive. All preferences are common knowledge. In a Match an individual from a set is paired up with one from another set i.e. ($i_N$, $j_N$). A matching is denoted with μ. For example, $\mu(i_N)=j_N$ and $\mu(j_N)=i_N$ The active Side are the proposers of a match, while the passive side respond to the active side's proposal for a match. A match is stable if every individual is matched with an acceptable match and there is no incentive for them to deviate from this match, as it is their best possible option. A pair of individuals from different sets that prefer each other to who they are paired with are a blocking pair. Blocking Pairs produces unstable matches. To solve a matching problem, we use the deferred acceptance algorithm. It is a Simple way to get rid of blocking pairs and to produce stable matches. For this algorithm, the passive side holds judgment on their match until the end of the algorithm.

# Definitions

**Active Side**: Proposers of a match

**Blocking Pair**: A pair of individuals from different sets that prefer each other to who they are paired with.

**Core**: no player can deviate from the "match" without disrupting the benefit it has to all players. It is a stable match.

**Deferred Acceptance**: The matching principle where the passive side holds judgment on their match until the end of the algorithm.

**Match**: An individual from a set that is paired up with one from another set. i.e. (M,W).

**Set**: A group of individuals who want to match with individuals in another group. i.e. I and J

**Stable Match:** i. Every individual is being matched with an acceptable partner. ii. There is no pair, each of whom would prefer to be matched with another rather than who they are paired with. If there is such a pair that does not follow these rules, then this pair is a blocking pair and it would be considered unstable.

**Passive Side**: Either accept or reject the active side's proposal. In the case of deferred acceptance, they can also retain.

**Proposal**: An offer made by one of the individuals on the active side to an agent from another set.

**Preference Table**: A table listing the preferences of each agent.

# End of Chapter Review Questions

#1

Goal: Find stable marriages for every individual

Two Sets: Males and Females

M= {Ethan, Fred, George, Henry}

F= {Alice, Beth, Carly, Danielle}

$M^N \succ_F F$ and $F^N \succ_M M$

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Alice | G | E | F | H |
| Beth | E | H | G | F |
| Carly | F | H | G | E |
| Danielle | H | F | E | G |
| Ethan | A | D | C | B |
| Fred | A | B | C | D |
| George | B | D | C | A |
| Henry | C | A | B | D |

Suppose the males are the active side. Find the stable matches.

#2

Using the information given in the previous example find the stable matches if females were the active side.

#3

Suppose there are medical students looking for jobs at hospitals. These hospitals are also currently looking for doctors to fill their positions. Since the hospitals are the employers in this situation, they will be the ones making offers to the medical students. The hospitals are the active side. There are three doctors in this market and three hospitals. Doctors={D1, D2, D3}

Hospitals={H1, H2, H3}

Preferences are as follows:

| | 1 | 2 | 3 |
|---|---|---|---|
| D1: | H1 | H2 | H3 |
| D2: | H1 | H2 | H3 |
| D3: | H1 | H3 | H2 |
| H1: | D1 | D2 | D3 |
| H2: | D1 | D3 | D2 |
| H3: | D1 | D2 | D3 |

Using this information, find the resulting matches.

#4

Using the same preferences as the problem above, find the resulting matches when the doctors are the active side of the market.

#5

Suppose there are two types of players I and J and within these categories there are 3 players I={$i_1, i_2, i_3$} J={$j_1, j_2, j_3$}. The preferences of each of the 6 players are as follows: P($i_1$)=$j_1, j_2, j_3$ P($i_2$)=$j_2, j_1, j_3$ P($i_3$)=$j_2, j_3, j_1$ P($j_1$)=$i_1, i_2, i_3$ P($j_2$)=$i_1, i_2, i_3$ P($j_3$)=$i_1, i_2, i_3$. Suppose in this scenario that the I's are the active side and propose a match to the J's who are the passive side. Draw up a Preference Table then find the matches that all of these criteria will lead to.

# Answers to End of Chapter Review Questions

#1
(A, E)
(B, H)
(C, F)
(D, G)

#2
(E, B)
(F, C)
(G, A)
(H, D)

#3
The first hospital H1 will make an offer to the first doctor [D1], which they will accept because H1 is D1s first choice.
μ(H1)=D1, μ(D1)=H1
H1 → D1

The second hospital [H2] will make an offer to their second choice because their first choice D1 is matched with H1 and has no incentive to switch. H2 will be matched with D3.
μ(H2)=D3, μ(D3)=H2
H2 → D3

The third hospital [H3] will make an offer to their second choice as well because D1 is already matched. The result is H3 is matched with D2.
μ(H3)=D2, μ(D2)=H3
H3 → D2

Resultant: (H1, D1) (H2, D3) (H3, H2)

#4
When the doctors make the offers, they all prefer H1.

D1 will make an offer to H1 which will accepts D1 (their first choice).
μ(H1)=D1, μ(D1)=H1
D1 → H1

D2 will then make an offer to H2 and D3 will make an offer to H3. They will both accept their offers.
μ(H2)=D2, μ(D2)=H2, μ(H3)=D3, μ(D3)=H3
D2 → H2 D3 → H3

Resultant: (H1, D1) (H2, D2) (H3, D3)

#5

| P | 1 | 2 | 3 |
|---|---|---|---|
| $i_1$ | $j_1$ | $j_2$ | $j_3$ |
| $i_2$ | $j_2$ | $j_1$ | $j_3$ |
| $i_3$ | $j_2$ | $j_3$ | $j_1$ |
| $j_1$ | $i_1$ | $i_2$ | $i_3$ |
| $j_2$ | $i_1$ | $i_2$ | $i_3$ |
| $j_3$ | $i_1$ | $i_2$ | $i_3$ |

1st proposal:
$i_1$ -> $j_1$
$i_2$ -> $j_2$

$i_3$ -> $j_2$

Results of 1st proposal:

$j_1$ retains $i_1$
$j_2$ rejects $i_3$
$j_2$ retains $i_2$

2nd proposal ($i_3$ is the only one who must change their offer)

$i_1$ -> $j_1$
$i_2$ -> $j_2$
$i_3$ -> $j_3$

Results of 2nd proposal:

$j_1$ retains $i_1$
$j_2$ retains $i_2$
$j_3$ retains $i_3$

End of algorithm, matches are as follows

$(i_1, j_1)$ $(i_2, j_2)$ $(i_3, j_3)$